CCSS REGULARITY Solve each equation if $0° \leq \theta \leq 360°$.

1. $2\sin\theta + 1 = 0$

SOLUTION:

$$\begin{aligned} 2\sin\theta + 1 &= 0 \\ 2\sin\theta &= -1 \\ \sin\theta &= \frac{-1}{2} \\ \theta &= 210° \text{ or } 330° \end{aligned}$$

The solutions are $210°, 330°$.

2. $\cos^2\theta + 2\cos\theta + 1 = 0$

SOLUTION:

$\cos^2\theta + 2\cos\theta + 1 = 0$

Let $x = \cos\theta$.

$$\begin{aligned} x^2 + 2x + 1 &= 0 \\ x^2 + x + x + 1 &= 0 \\ x(x+1) + 1(x+1) &= 0 \\ (x+1)(x+1) &= 0 \end{aligned}$$

$x = -1$

$$\begin{aligned} \cos\theta &= -1 \\ \theta &= 180° \end{aligned}$$

The solution is $180°$.

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3. $\cos 2\theta + \cos\theta = 0$

SOLUTION:

$$\begin{aligned} \cos 2\theta + \cos\theta &= 0 \\ \left(2\cos^2\theta - 1\right) + \cos\theta &= 0 \\ 2\cos^2\theta + \cos\theta - 1 &= 0 \\ 2\cos^2\theta + 2\cos\theta - \cos\theta - 1 &= 0 \\ 2\cos\theta(\cos\theta + 1) - 1(\cos\theta + 1) &= 0 \\ (\cos\theta + 1)(2\cos\theta - 1) &= 0 \end{aligned}$$

$$\begin{aligned} (\cos\theta + 1) = 0 \text{ or } (2\cos\theta - 1) &= 0 \\ \cos\theta = -1 \text{ or } 2\cos\theta &= 1 \\ \theta = 180° \text{ or } \cos\theta &= \frac{1}{2} \\ \theta = 180° \text{ or } \theta &= 60° \text{ or } 300° \end{aligned}$$

The solutions are $60°, 180°, 300°$.

4. $2\cos\theta = 1$

SOLUTION:

$$\begin{aligned} 2\cos\theta &= 1 \\ \cos\theta &= \frac{1}{2} \\ \theta &= 60° \text{ or } 300° \end{aligned}$$

The solutions are $60°, 300°$.

5. $\cos\theta = -\frac{\sqrt{3}}{2}$

SOLUTION:

$$\cos\theta = -\frac{\sqrt{3}}{2}$$
$$\theta = 150° \text{ or } \theta = 210°$$

The solutions are $150°, 210°$.

6. $\sin 2\theta = -\frac{\sqrt{3}}{2}$

SOLUTION:

$$\begin{aligned}\sin 2\theta &= -\frac{\sqrt{3}}{2}\\ 2\sin\theta\cos\theta &= -\frac{\sqrt{3}}{2}\\ \sin\theta\cos\theta &= -\frac{\sqrt{3}}{4}\\ &= \left(-\frac{1}{2}\right)\cdot\frac{\sqrt{3}}{2}\text{ or }\frac{\sqrt{3}}{2}\cdot\left(-\frac{1}{2}\right)\end{aligned}$$

The solutions are $120°, 150°, 300°, 330°$.

7. $\cos 2\theta = 8 - 15\sin\theta$

SOLUTION:

$$\begin{aligned}\cos 2\theta &= 8 - 15\sin\theta\\ 1 - 2\sin^2\theta &= 8 - 15\sin\theta\\ -2\sin^2\theta &= 8 - 15\sin\theta - 1\\ -2\sin^2\theta &= 7 - 15\sin\theta\end{aligned}$$

$$\begin{aligned}-2\sin^2\theta + 15\sin\theta - 7 &= 0\\ 2\sin^2\theta - 15\sin\theta + 7 &= 0\\ 2\sin^2\theta - 14\sin\theta - \sin\theta + 7 &= 0\\ 2\sin\theta(\sin\theta - 7) - (\sin\theta - 7) &= 0\\ (\sin\theta - 7)(2\sin\theta - 1) &= 0\end{aligned}$$

$$\begin{aligned}\sin\theta - 7 &= 0\\ \sin\theta &= 7,\text{ Invalid value}\end{aligned} \quad\text{or}\quad \begin{aligned}2\sin\theta - 1 &= 0\\ 2\sin\theta &= 1\\ \sin\theta &= \frac{1}{2}\\ \theta &= 30°\text{ or}\\ \theta &= 150°\end{aligned}$$

The solutions are $30°, 150°$.

8. $\sin\theta + \cos\theta = 1$

SOLUTION:

$$\sin\theta + \cos\theta = 1$$

$$\sqrt{\frac{1-\cos 2\theta}{2}} + \sqrt{\frac{1+\cos 2\theta}{2}} = 1$$

$$\sqrt{1-\cos 2\theta} + \sqrt{1+\cos 2\theta} = \sqrt{2}$$

$$\left(\sqrt{1-\cos 2\theta} + \sqrt{1+\cos 2\theta}\right)^2 = 2$$

$$(1-\cos 2\theta) + 2\left(\sqrt{1-\cos 2\theta}\right)\left(\sqrt{1+\cos 2\theta}\right) + (1+\cos 2\theta) = 2$$

$$2-\cos 2\theta + 2\left(\sqrt{1-\cos^2 2\theta}\right) + \cos 2\theta - 2 = 0$$

$$2\left(\sqrt{1-\cos^2 2\theta}\right) = 0$$

$$\sqrt{1-\cos^2 2\theta} = 0$$

$$\sqrt{\sin^2 2\theta} = 1$$

$$\sin 2\theta = 0$$

$$\theta = 0, \frac{\pi}{2}, \cancel{\pi}, \cancel{\frac{3\pi}{2}}, 2\pi$$

π and $\frac{3\pi}{2}$ are extraneous solutions when checked with the original equation.

The solutions are $0°, 90°, 360°$.

Solve each equation for all values of θ if θ is measured in radians.

9. $4\sin^2\theta - 1 = 0$

SOLUTION:

$$\begin{aligned} 4\sin^2\theta - 1 &= 0 \\ 4\sin^2\theta &= 1 \\ \sin^2\theta &= \frac{1}{4} \\ \sin\theta &= \pm\frac{1}{2} \end{aligned}$$

$$\sin\theta = \frac{1}{2} \qquad \sin\theta = -\frac{1}{2}$$
$$\theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \qquad \theta = -\frac{\pi}{6} \text{ or } -\frac{5\pi}{6}$$

Let T be the period.

$$T = 2\pi$$

So, the general solution is $\theta = \pm\frac{\pi}{6} + 2k\pi$ or $\pm\frac{5\pi}{6} + 2k\pi$, where k is an integer.

10. $2\cos^2\theta = 1$

SOLUTION:

$$\begin{aligned} 2\cos^2\theta &= 1 \\ \cos^2\theta &= \frac{1}{2} \\ \cos\theta &= \pm\frac{1}{\sqrt{2}} \end{aligned}$$

$$\cos\theta = \frac{1}{\sqrt{2}} \qquad \cos\theta = -\frac{1}{\sqrt{2}}$$
$$\theta = \frac{\pi}{4} \qquad \theta = \frac{3\pi}{4}$$

$$\frac{3\pi}{4} - \frac{\pi}{4} = \frac{2\pi}{4} = \frac{\pi}{2}$$

The period for this problem is $\frac{\pi}{2}$.

So, the general solution is $\frac{\pi}{4} + \frac{k}{2}\pi$.

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11. $\cos 2\theta \sin\theta = 1$

SOLUTION:

$$\begin{aligned} \cos 2\theta \sin\theta &= 1 \\ \left(1 - 2\sin^2\theta\right)\sin\theta &= 1 \\ \sin\theta - 2\sin^3\theta - 1 &= 0 \\ 2\sin^3\theta - \sin\theta + 1 &= 0 \end{aligned}$$

Let $x = \sin\theta$. So, the equation can be written as $2x^3 - x + 1 = 0$.
The above equation has one real root, –1, and two imaginary roots.

Therefore, $\begin{aligned} \sin\theta &= -1 \\ \theta &= \frac{3\pi}{2}, \frac{7\pi}{2} \end{aligned}$.

$$\frac{7\pi}{2} - \frac{3\pi}{2} = \frac{4\pi}{2} = 2\pi$$

The period for this problem is 2π.

So, the general solution is $\frac{3\pi}{2} + 2k\pi$, where k is an integer.

12. $\sin\frac{\theta}{2} + \cos\frac{\theta}{2} = \sqrt{2}$

SOLUTION:

$$\begin{aligned}
\sin\frac{\theta}{2} + \cos\frac{\theta}{2} &= \sqrt{2} \\
\sqrt{\frac{1-\cos\theta}{2}} + \sqrt{\frac{1+\cos\theta}{2}} &= \sqrt{2} \\
\sqrt{1-\cos\theta} + \sqrt{1+\cos\theta} &= 2 \\
\left(\sqrt{1-\cos\theta} + \sqrt{1+\cos\theta}\right)^2 &= 4 \\
(1-\cos\theta) + 2\left(\sqrt{1-\cos\theta}\right)\left(\sqrt{1+\cos\theta}\right) + (1+\cos\theta) &= 4 \\
2 - \cos\theta + 2(\sqrt{1-\cos^2\theta}) + \cos\theta - 4 &= 0 \\
2(\sqrt{1-\cos^2\theta}) - 2 &= 0 \\
\sqrt{1-\cos^2\theta} &= 1 \\
1-\cos^2\theta &= 1 \\
\cos\theta &= 0 \\
\theta &= \frac{\pi}{2}, \frac{9\pi}{2}
\end{aligned}$$

$$\frac{9\pi}{2} - \frac{\pi}{2} = \frac{8\pi}{2} = 4\pi$$

The period for this problem is 4π.

So, the general solution is $\frac{\pi}{2} + 4\pi k$, where k is an integer.

13. $\cos 2\theta + 4\cos\theta = -3$

SOLUTION:

$$\begin{aligned}
\cos 2\theta + 4\cos\theta &= -3 \\
\left(2\cos^2\theta - 1\right) + 4\cos\theta + 3 &= 0 \\
2\cos^2\theta - 1 + 4\cos\theta + 3 &= 0 \\
2\cos^2\theta + 4\cos\theta + 2 &= 0 \\
\cos^2\theta + 2\cos\theta + 1 &= 0 \\
(\cos\theta + 1)(\cos\theta + 1) &= 0 \\
(\cos\theta + 1)^2 &= 0 \\
\cos\theta + 1 &= 0 \\
\cos\theta &= -1 \\
\theta &= \pi, 3\pi
\end{aligned}$$

$3\pi - \pi = 2\pi$

The period for this problem is 2π.
So, the general solution is $\pi + 2k\pi$, where k is an integer.

14. $\sin\frac{\theta}{2}+\cos\theta=1$

SOLUTION:

$$\begin{aligned}
\sin\frac{\theta}{2}+\cos\theta&=1\\
\sin\frac{\theta}{2}&=1-\cos\theta\\
\sin\frac{\theta}{2}&=1-\left(1-2\sin^2\frac{\theta}{2}\right)\\
\sin\frac{\theta}{2}&=1-1+2\sin^2\frac{\theta}{2}\\
\sin\frac{\theta}{2}&=2\sin^2\frac{\theta}{2}\\
\sin\frac{\theta}{2}-2\sin\frac{\theta}{2}&=0\\
\sin\frac{\theta}{2}\left(1-2\sin\frac{\theta}{2}\right)&=0\\
\sin\frac{\theta}{2}=0 \quad &\text{or} \quad 1-2\sin\frac{\theta}{2}=0\\
\frac{\theta}{2}=0 \quad &\text{or} \quad \sin\frac{\theta}{2}=\frac{1}{2}\\
\theta=0 \quad &\text{or} \quad \theta=\frac{\pi}{3},\ \frac{5\pi}{3}
\end{aligned}$$

Let T be the period.

$$\frac{T}{2}=2\pi$$
$$T=4\pi$$

So, the period of the function is 4π.

So, the general solution is $0+2k\pi;\frac{\pi}{3}+4k\pi;\frac{5\pi}{3}+4k\pi$.

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Solve each equation for all values of θ if θ is measured in degrees.

15. $\cos 2\theta - \sin^2\theta + 2 = 0$

SOLUTION:

$$\begin{aligned}\cos 2\theta - \sin^2\theta + 2 &= 0\\ 1 - 2\sin^2\theta - \sin^2\theta + 2 &= 0\\ 3 - 3\sin^2\theta &= 0\\ 1 - \sin^2\theta &= 0\\ \sin^2\theta &= 1\\ \sin\theta &= \pm 1\\ \theta &= 90°, 270°\end{aligned}$$

$270° - 90° = 180°$

The period for this problem is $180°$.
So, the general solution is $90° + k \cdot 180°$, where k is an integer.

16. $\sin^2\theta - \sin\theta = 0$

SOLUTION:

$$\begin{aligned}\sin^2\theta - \sin\theta &= 0\\ \sin\theta(\sin\theta - 1) &= 0\end{aligned}$$

$$\begin{aligned}\sin\theta &= 0\\ \theta &= 0° \text{ or } 180° \text{ or } 360°\end{aligned}$$

$$\begin{aligned}\sin\theta - 1 &= 0\\ \sin\theta &= 1\\ \theta &= 90°\end{aligned}$$

The general solution is $0° + k \cdot 180°, 90° + k \cdot 360°$, where k is an integer.

17. $2\sin^2\theta - 1 = 0$

SOLUTION:

$$\begin{aligned} 2\sin^2\theta - 1 &= 0 \\ 2\sin^2\theta &= 1 \\ \sin^2\theta &= \frac{1}{2} \\ \sin\theta &= \pm\frac{1}{\sqrt{2}} \\ \theta &= 45°, 135° \end{aligned}$$

$135° - 45° = 90°$

The period for this problem is $90°$.
So, the general solution is $45° + k \cdot 90°$, where k is an integer.

18. $\cos\theta - 2\cos\theta\sin\theta = 0$

SOLUTION:

$$\begin{aligned} \cos\theta - 2\cos\theta\sin\theta &= 0 \\ \cos\theta(1 - 2\sin\theta) &= 0 \end{aligned}$$

$$\begin{aligned} \cos\theta &= 0 \\ \theta &= 90°, 270° \end{aligned} \qquad \begin{aligned} 1 - 2\sin\theta &= 0 \\ -2\sin\theta &= -1 \\ \sin\theta &= \frac{1}{2} \\ \theta &= 30°, 150° \end{aligned}$$

The general solution is $30° + k \cdot 360°$, $150° + k \cdot 360°$, $90° + k \cdot 180°$.

19. $\cos 2\theta \sin\theta = 1$

SOLUTION:

$$\begin{aligned} \cos 2\theta \sin\theta &= 1 \\ (1 - 2\sin^2\theta)\sin\theta &= 1 \\ \sin\theta - 2\sin^3\theta - 1 &= 0 \\ 2\sin^3\theta - \sin\theta + 1 &= 0 \end{aligned}$$

Let $x = \sin\theta$. So, the equation can be written as $2x^3 - x + 1 = 0$.
The above equation has one real root, –1 and two imaginary roots.

$$\begin{aligned} \sin\theta &= -1 \\ \theta &= 270°, 630°. \end{aligned}$$

So, the general solution is $270° + k \cdot 360°$, where k is an integer.

20. $\sin\theta\tan\theta - \tan\theta = 0$

SOLUTION:

$$\sin\theta\tan\theta - \tan\theta = 0$$
$$\tan\theta(\sin\theta - 1) = 0$$

$$\begin{aligned} \tan\theta &= 0 \\ \theta &= 0^\circ \end{aligned} \qquad \begin{aligned} \sin\theta - 1 &= 0 \\ \sin\theta &= 1 \\ \theta &= 90^\circ \end{aligned}$$

The general solution is $0^\circ + k \cdot 180^\circ, 90^\circ + k \cdot 360^\circ$, where k is an integer.

21. **LIGHT** The number of hours of daylight d in Hartford, Connecticut, may be approximated by the equation $d = 3\sin\frac{2\pi}{365}t + 12$, where t is the number of days after March 21.

a. On what days will Hartford have exactly $10\frac{1}{2}$ hours of daylight?

b. Using the results in part **a**, tell what days of the year have at least $10\frac{1}{2}$ hours of daylight. Explain how you know.

SOLUTION:

a. $d = 3\sin\frac{2\pi}{365}t + 12$

Substitute $10\frac{1}{2}$ for *d* in the above equation.

$$\begin{aligned} 10\frac{1}{2} &= 3\sin\frac{2\pi}{365}t + 12 \\ \frac{21}{2} - 12 &= 3\sin\frac{2\pi}{365}t \\ -\frac{3}{2} &= 3\sin\frac{2\pi}{365}t \\ -\frac{1}{2} &= \sin\frac{2\pi}{365}t \\ \frac{2\pi}{365}t &= \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

$$\begin{aligned} \frac{2\pi}{365}t &= \frac{7\pi}{6} \\ t &= \frac{7\pi}{6}\cdot\frac{365}{2\pi} \\ t &\approx 213 \end{aligned} \qquad \begin{aligned} \frac{2\pi}{365}t &= \frac{11\pi}{6} \\ t &= \frac{11\pi}{6}\cdot\frac{365}{2\pi} \\ t &\approx 335 \end{aligned}$$

There will be $10\frac{1}{2}$ hours of daylight 213 and 335 days after March 21.

After March 21, 213^{th} day is October 20 and 335^{th} day is February 19.

b. $$\begin{aligned} d &\geq 3\sin\frac{2\pi}{365}t + 12 \\ t &\leq 213 \text{ and } t \leq 335 \end{aligned}$$

Since the longest day of the year occurs around June 22, the days between February 19 and October 20 must increase in length until June 22 and then decrease in length until October 20.

Solve each equation.

22. $\sin^2 2\theta + \cos^2 \theta = 0$

SOLUTION:

$$\begin{aligned} \sin^2 2\theta + \cos^2 \theta &= 0 \\ (2\sin\theta\cos\theta)^2 + \cos^2\theta &= 0 \\ 4\sin^2\theta\cos^2\theta + \cos^2\theta &= 0 \\ \cos^2\theta(4\sin^2\theta + 1) &= 0 \end{aligned}$$

$$\begin{aligned} \cos^2\theta &= 0 \\ \cos\theta &= 0 \\ \theta &= \frac{\pi}{2}, \frac{3\pi}{2} \end{aligned}$$

$$\begin{aligned} 4\sin^2\theta + 1 &= 0 \\ 4\sin^2\theta &= -1 \\ 4\sin^2\theta &= -\frac{1}{4} \end{aligned}$$

$\sin^2\theta$ never gets negaive value.

$$\frac{3\pi}{2} - \frac{\pi}{2} = \frac{2\pi}{2} = \pi$$

So, the period of the function is π.

So, the general solution is $\frac{\pi}{2} + \pi k$, where k is an integer.

23. $\tan^2\theta - 2\tan\theta + 1 = 0$

SOLUTION:

$$\begin{aligned} \tan^2\theta - 2\tan\theta + 1 &= 0 \\ \tan^2\theta - \tan\theta - \tan\theta + 1 &= 0 \\ \tan\theta(\tan\theta - 1) - 1(\tan\theta - 1) &= 0 \\ (\tan\theta - 1)(\tan\theta - 1) &= 0 \\ (\tan\theta - 1)^2 &= 0 \\ \tan\theta - 1 &= 0 \\ \tan\theta &= 1 \\ \theta &= \frac{3\pi}{4}, \frac{7\pi}{4} \end{aligned}$$

$$\frac{7\pi}{4} - \frac{3\pi}{4} = \frac{4\pi}{4} = \pi$$

The period for tan function is π.

So, the general solution is $\frac{3\pi}{4} + \pi k$, k is an integer.

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24. $\cos^2\theta + 3\cos\theta = -2$

SOLUTION:

$$\begin{aligned}\cos^2\theta + 3\cos\theta &= -2\\ \cos^2\theta + 3\cos\theta + 2 &= 0\\ \cos^2\theta + \cos\theta + 2\cos\theta + 2 &= 0\\ \cos\theta(\cos\theta + 1) + 2(\cos\theta + 1) &= 0\\ (\cos\theta + 1)(\cos\theta + 2) &= 0\end{aligned}$$

$$\begin{array}{rl}\cos\theta + 1 = 0 & \cos\theta + 2 = 0\\ \cos\theta = -1 & \cos\theta = -2\\ \theta = \pi, 3\pi & \text{Invalid input function value}\end{array}$$

$$3\pi - \pi = 2\pi$$

The period for cosine function is 2π.
So, the general solution is $\pi + 2\pi k$, k is an integer.

25. $\sin 2\theta - \cos\theta = 0$

SOLUTION:

$$\begin{aligned}\sin 2\theta - \cos\theta &= 0\\ 2\sin\theta\cos\theta - \cos\theta &= 0\\ \cos\theta(2\sin\theta - 1) &= 0\end{aligned}$$

$$\begin{array}{ll} & 2\sin\theta - 1 = 0\\ \cos\theta = 0 & 2\sin\theta = 1\\ \theta = \frac{\pi}{2}, \frac{3\pi}{2} & \sin\theta = \frac{1}{2}\\ & \theta = \frac{\pi}{6}, \frac{5\pi}{6}\end{array}$$

The general solution is $\frac{\pi}{2} + \pi k, \frac{\pi}{6} + 2\pi k, \frac{5\pi}{6} + 2\pi k$, where k is an integer.

26. $\tan\theta = 1$

SOLUTION:

$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4}, \frac{5\pi}{4} \text{ or } 45°, 225°$$

$$\frac{5\pi}{4} - \frac{\pi}{4} = \frac{4\pi}{4} = \pi$$

The period for tan function is π.

So, the general solution is $45° + k \cdot 180° \, or \, \frac{\pi}{4} + k \cdot \pi$, k is an integer.

27. $\cos 8\theta = 1$

SOLUTION:

$$\cos 8\theta = 1$$
$$8\theta = 0, 2\pi$$
$$\theta = 0, \frac{\pi}{4}$$

$$\frac{\pi}{4} - 0 = \frac{\pi}{4}$$

The period for tan function is $\frac{\pi}{4}$.

So, the general solution is $0° + k \cdot 45° \, or \, 0° + k \cdot \frac{\pi}{4}$, k is an integer.

28. $\sin\theta + 1 = \cos 2\theta$

SOLUTION:

$$\sin\theta + 1 = \cos 2\theta$$
$$\sin\theta + 1 = 1 - 2\sin^2\theta$$
$$\sin\theta + 2\sin^2\theta = 0$$
$$\sin\theta(1 + 2\sin\theta) = 0$$

$$\sin\theta = 0$$
$$\theta = 0°, 180°$$

$$1 + 2\sin\theta = 0$$
$$2\sin\theta = -1$$
$$\sin\theta = -\frac{1}{2}$$
$$\theta = 210°, 330°$$

The general solution is $0° + k \cdot 180°, 210° + k \cdot 360°, 330° + k \cdot 360°$, k is an integer.

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29. $2\cos^2\theta = \cos\theta$

SOLUTION:

$$\begin{aligned} 2\cos^2\theta &= \cos\theta \\ 2\cos^2\theta - \cos\theta &= 0 \\ \cos\theta(\cos\theta - 1) &= 0 \end{aligned}$$

$$\begin{array}{lll} \cos\theta = 0 & \text{or} & 2\cos\theta - 1 = 0 \\ \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} & & 2\cos\theta = 1 \\ & & \cos\theta = \frac{1}{2} \\ & & \theta = \frac{\pi}{3} \text{ or } \frac{5\pi}{3} \end{array}$$

Since cosine has a period of 2π, the solutions on the interval (–∞, ∞) have the general form $x = \frac{\pi}{3} + 2k\pi$, $x = \frac{\pi}{2} + k\pi$, and $x = \frac{5\pi}{3} + 2k\pi$, where k is an integer.

Solve each equation for the given interval.

30. $\cos^2\theta = \frac{1}{4};\ 0° \le \theta \le 360°$

SOLUTION:

$$\begin{aligned} \cos^2\theta &= \frac{1}{4} \\ \cos\theta &= \pm\frac{1}{2} \end{aligned}$$

$$\begin{array}{ll} \cos\theta = \frac{1}{2} & \cos\theta = -\frac{1}{2} \\ \theta = 60°, 300° & \theta = 120°, 240° \end{array}$$

The solutions are $60°, 120°, 240°, 300°$.

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31. $2\sin^2\theta = 1;\ 90° < \theta < 270°$

SOLUTION:

$$2\sin^2\theta = 1$$
$$\sin^2\theta = \frac{1}{2}$$
$$\sin\theta = \pm\frac{1}{\sqrt{2}}$$

$\sin\theta = \frac{1}{\sqrt{2}}$ $\quad \sin\theta = -\frac{1}{\sqrt{2}}$, where $90° < \theta < 270°$.

$\theta = 135° \quad \theta = 225°$

32. $\sin 2\theta - \cos\theta = 0;\ 0 \le \theta \le 2\pi$

SOLUTION:

$$\sin 2\theta - \cos\theta = 0$$
$$2\sin\theta\cos\theta - \cos\theta = 0$$
$$\cos\theta(2\sin\theta - 1) = 0$$
$$\cos\theta = 0 \text{ or } 2\sin\theta - 1 = 0$$
$$\theta = \frac{\pi}{2}, \frac{3\pi}{3} \text{ or } \sin\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

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33. $3\sin^2\theta = \cos^2\theta;\ 0 \le \theta \le \frac{\pi}{2}$

SOLUTION:

$$\begin{aligned}
3\sin^2\theta &= \cos^2\theta \\
3\sin^2\theta - \cos^2\theta &= 0 \\
3\left(1-\cos^2\theta\right) - \cos^2\theta &= 0 \\
3 - 3\cos^2\theta - \cos^2\theta &= 0 \\
3 - 4\cos^2\theta &= 0 \\
-4\cos^2\theta &= -3 \\
4\cos^2\theta &= 3 \\
\cos^2\theta &= \frac{3}{4} \\
\cos\theta &= \pm\frac{\sqrt{3}}{2}
\end{aligned}$$

$$\begin{array}{ll}
\cos\theta = \frac{\sqrt{3}}{2} & \cos\theta = -\frac{\sqrt{3}}{2} \\
\theta = \frac{\pi}{6} & \theta = \frac{5\pi}{6}
\end{array}$$

Since $0 \le \theta \le \frac{\pi}{2}$, the solution is $\frac{\pi}{6}$.

34. $2\sin\theta + \sqrt{3} = 0;\ 180° < \theta < 360°$

SOLUTION:

$$\begin{aligned}
2\sin\theta + \sqrt{3} &= 0 \\
2\sin\theta &= -\sqrt{3} \\
\sin\theta &= -\frac{\sqrt{3}}{2} \\
\theta &= 240°, 300°
\end{aligned}$$

The solutions are $240°, 300°$.

35. $4\sin^2\theta - 1 = 0;\ 180° < \theta < 360°$

SOLUTION:

$$\begin{aligned} 4\sin^2\theta - 1 &= 0 \\ 4\sin^2\theta &= 1 \\ \sin^2\theta &= \frac{1}{4} \\ \sin\theta &= \pm\frac{1}{2} \end{aligned}$$

$$\begin{aligned} \sin\theta &= \frac{1}{2} & \qquad \sin\theta &= -\frac{1}{2} \\ \theta &= 30°, 150° & \qquad \theta &= 210°, 330° \end{aligned}$$

Since $180° < \theta < 360°$, the solutions are $210°, 330°$.

Solve each equation for all values of θ if θ is measured in radians

36. $\cos 2\theta + 3\cos\theta = 1$

SOLUTION:

$$\begin{aligned} \cos 2\theta + 3\cos\theta &= 1 \\ \left(2\cos^2\theta - 1\right) + 3\cos\theta &= 1 \\ 2\cos^2\theta - 1 + 3\cos\theta - 1 &= 0 \\ 2\cos^2\theta + 3\cos\theta - 2 &= 0 \\ (\cos\theta + 2)(2\cos\theta - 1) &= 0 \end{aligned}$$

$$\cos\theta = -2 \ \text{ or } \ \cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

Since the range of the cosine function lies between –1 and 1, neglect the equation $\cos\theta = -2$.

The general solution is $\frac{\pi}{3} + 2k\pi,\ \frac{5\pi}{3} + 2k\pi$, k is an integer.

37. $2\sin^2\theta = \cos\theta + 1$

SOLUTION:

$$\begin{aligned}
2\sin^2\theta &= \cos\theta + 1 \\
2\left(1-\cos^2\theta\right) &= \cos\theta + 1 \\
2-2\cos^2\theta-\cos\theta-1 &= 0 \\
-2\cos^2\theta-\cos\theta+1 &= 0 \\
2\cos^2\theta+\cos\theta-1 &= 0 \\
2\cos^2\theta+2\cos\theta-\cos\theta-1 &= 0 \\
2\cos\theta(\cos\theta+1)-1(\cos\theta+1) &= 0 \\
(\cos\theta+1)(2\cos\theta-1) &= 0
\end{aligned}$$

$$\cos\theta = -1 \text{ or } \cos\theta = \frac{1}{2}$$

$$\theta = \pi \quad \text{or} \quad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

The general solution is $\pi + 2k\pi, \frac{\pi}{3} + 2k\pi, \frac{5\pi}{3} + 2k\pi$, where k is an integer.

38. $\cos^2\theta - \frac{3}{2} = \frac{5}{2}\cos\theta$

SOLUTION:

$$\begin{aligned}
\cos^2\theta - \frac{3}{2} &= \frac{5}{2}\cos\theta \\
\frac{2\cos^2\theta - 3}{2} &= \frac{5}{2}\cos\theta \\
2\cos^2\theta - 3 &= 5\cos\theta \\
2\cos^2\theta - 5\cos\theta - 3 &= 0 \\
2\cos^2\theta - 6\cos\theta + \cos\theta - 3 &= 0 \\
2\cos\theta(\cos\theta - 3) + 1(\cos\theta - 3) &= 0 \\
(\cos\theta - 3)(2\cos\theta + 1) &= 0
\end{aligned}$$

$$\begin{aligned}
\cos\theta - 3 &= 0 \\
\cos\theta &= 3
\end{aligned}$$

Invalid input function value

$$\begin{aligned}
2\cos\theta + 1 &= 0 \\
2\cos\theta &= -1 \\
\cos\theta &= -\frac{1}{2} \\
\theta &= \frac{2\pi}{3}, \frac{4\pi}{3}
\end{aligned}$$

The period for the cosine function is 2π.

So, the general solution is $\frac{2\pi}{3} + 2k\pi, \frac{4\pi}{3} + 2k\pi$, where k is an integer.

39. $3\cos\theta - \cos\theta = 2$

SOLUTION:

$$\begin{aligned}
3\cos\theta - \cos\theta &= 2 \\
2\cos\theta &= 2 \\
\cos\theta &= 1 \\
\theta &= 0, 2\pi
\end{aligned}$$

$2\pi - 0 = 2\pi$

The period for this given function is 2π.
So, the general solution is $0 + 2k\pi$, where k is an integer.

13-5 Solving Trigonometric Equations

Solve each equation for all values of θ if θ is measured in degrees

40. $\sin\theta - \cos\theta = 0$

SOLUTION:

$$\begin{aligned} \sin\theta - \cos\theta &= 0 \\ \sin\theta &= \cos\theta \\ \theta &= 45°, 225° \end{aligned}$$

$225° - 45° = 180°$

The period for this given function is $180°$.
So, the general solution is $45° + k \cdot 180°$, where k is an integer.

41. $\tan\theta - \sin\theta = 0$

SOLUTION:

$$\begin{aligned} \tan\theta - \sin\theta &= 0 \\ \sin\theta &= \tan\theta \\ \theta &= 0°, 180° \end{aligned}$$

$180° - 0° = 180°$

The period for this given function is $180°$.
So, the general solution is $0° + k \cdot 180°$, where k is an integer.

42. $\sin^2\theta = 2\sin\theta + 3$

SOLUTION:

$$\begin{aligned} \sin^2\theta &= 2\sin\theta + 3 \\ \sin^2\theta - 2\sin\theta - 3 &= 0 \\ \sin^2\theta + \sin\theta - 3\sin\theta - 3 &= 0 \\ \sin\theta(\sin\theta + 1) - 3(\sin\theta + 1) &= 0 \\ (\sin\theta + 1)(\sin\theta - 3) &= 0 \end{aligned}$$

$$\begin{array}{ll} \sin\theta + 1 = 0 & \sin\theta - 3 = 0 \\ \sin\theta = -1 & \sin\theta = 3 \\ \theta = 270° & \text{Invalid input fucntion value} \end{array}$$

The period for the sine function is $360°$.
So, the general solution is $270° + k \cdot 360°$, where k is an integer.

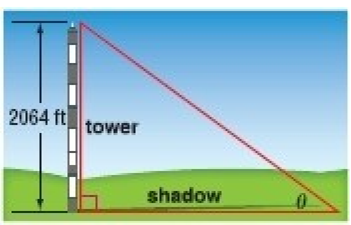
43. $4\sin^2 = 4\sin\theta - 1$

SOLUTION:

$$\begin{aligned}
4\sin^2\theta &= 4\sin\theta - 1\\
4\sin^2\theta - 4\sin\theta + 1 &= 0\\
4\sin^2\theta - 2\sin\theta - 2\sin\theta + 1 &= 0\\
2\sin\theta(2\sin\theta - 1) - 1(2\sin\theta - 1) &= 0\\
(2\sin\theta - 1)(2\sin\theta - 1) &= 0\\
(2\sin\theta - 1)^2 &= 0\\
2\sin\theta - 1 &= 0\\
2\sin\theta &= 1\\
\sin\theta &= \frac{1}{2}\\
\theta &= 30°, 150°
\end{aligned}$$

The period for the sine function is $360°$.
So, the general solution is $30° + k \cdot 360°, 150° + k \cdot 360°$, where k is an integer.

44. **ELECTRONICS** One of the tallest structures in the world is a television transmitting tower located near Fargo, North Dakota, with a height of 2064 feet. What is the measure of θ if the length of the shadow is 1 mile?



SOLUTION:
1 mile = 5280 ft
Opposite side = 2064 ft
Adjacent side = 5280 ft

$$\begin{aligned}
\tan\theta &= \frac{2064}{5280}\\
&= 0.39090\\
\theta &= \tan^{-1}(0.39090)\\
\theta &\approx 21°
\end{aligned}$$

13-5 Solving Trigonometric Equations

Solve each equation.

45. $2\sin^2\theta = 3\sin\theta + 2$

SOLUTION:

$$\begin{aligned} 2\sin^2\theta &= 3\sin\theta + 2 \\ 2\sin^2\theta - 3\sin\theta - 2 &= 0 \\ 2\sin^2\theta - 4\sin\theta + \sin\theta - 2 &= 0 \\ 2\sin\theta(\sin\theta - 2) + 1(\sin\theta - 2) &= 0 \\ (\sin\theta - 2)(2\sin\theta + 1) &= 0 \end{aligned}$$

$$\begin{aligned} \sin\theta - 2 &= 0 \\ \sin\theta &= 2 \end{aligned}$$

Invalid input function value

$$\begin{aligned} 2\sin\theta + 1 &= 0 \\ 2\sin\theta &= -1 \\ \sin\theta &= -\frac{1}{2} \\ \theta &= 210°, 330° \text{ or } \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

The period for the sine function is $360° \text{ or } 2\pi$.

So, the general solution is $\frac{7\pi}{6} + 2k\pi, \frac{11\pi}{6} + 2k\pi$ or $210° + k \cdot 360°, 330° + k \cdot 360°$, where k is an integer.

46. $2\cos^2\theta + 3\sin\theta = 3$

SOLUTION:

$$\begin{aligned} 2\cos^2\theta + 3\sin\theta &= 3 \\ 2(1 - \sin^2\theta) + 3\sin\theta &= 3 \\ 2 - 2\sin^2\theta + 3\sin\theta - 3 &= 0 \\ 2\sin^2\theta - 3\sin\theta + 1 &= 0 \\ 2\sin^2\theta - 2\sin\theta - \sin\theta + 1 &= 0 \\ 2\sin\theta(\sin\theta - 1) - 1(\sin\theta - 1) &= 0 \\ (\sin\theta - 1)(2\sin\theta - 1) &= 0 \end{aligned}$$

$$\begin{aligned} \sin\theta - 1 &= 0 \\ \sin\theta &= 1 \\ \theta &= \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} 2\sin\theta - 1 &= 0 \\ 2\sin\theta &= 1 \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

The general solution is $\frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi, \frac{\pi}{2} + 2k\pi$, k is an integer.

13-5 Solving Trigonometric Equations

47. $\sin^2\theta + \cos 2\theta = \cos\theta$

SOLUTION:

$$\begin{aligned} \sin^2\theta + \cos 2\theta &= \cos\theta \\ 1 - \cos^2\theta + 2\cos^2\theta - 1 - \cos\theta &= 0 \\ \cos^2\theta - \cos\theta &= 0 \\ \cos\theta(\cos\theta - 1) &= 0 \end{aligned}$$

$$\cos\theta = 0 \quad \text{or} \quad \cos\theta - 1 = 0$$
$$\theta = \frac{\pi}{2} \qquad \cos\theta = 1$$
$$\theta = 0$$

The general solution is $0 + 2k\pi, \frac{\pi}{2} + k\pi$, k is an integer.

48. $2\cos^2\theta = -\cos\theta$

SOLUTION:

$$\begin{aligned} 2\cos^2\theta &= -\cos\theta \\ 2\cos^2\theta + \cos\theta &= 0 \\ \cos\theta(2\cos\theta + 1) &= 0 \end{aligned}$$

$$2\cos\theta + 1 = 0$$
$$2\cos\theta = -1$$
$$\cos\theta = 0 \qquad \cos\theta = -\frac{1}{2}$$
$$\theta = 90^\circ \qquad \theta = 120^\circ, 240^\circ$$

The general solution is $90^\circ + k \cdot 180^\circ, 120^\circ + k \cdot 360^\circ, 240^\circ + k \cdot 360^\circ$, k is an inger.

49. **CCSS SENSE-MAKING** Due to ocean tides, the depth y in meters of the River Thames in London varies as a sine function of x, the hour of the day. On a certain day that function was $y = 3\sin\left[\frac{\pi}{6}(x-4)\right] + 8$, where x = 0, 1, 2, ..., 24 corresponds to 12:00 midnight, 1:00 A.M., 2:00 A.M., ..., 12:00 midnight the next night.

a. What is the maximum depth of the River Thames on that day?

b. At what times does the maximum depth occur?

SOLUTION:

a. For maximum depth, the derivative of y should be zero.

So,

13-5 Solving Trigonometric Equations

$$
\begin{aligned}
y &= 3\sin\left[\frac{\pi}{6}(x-4)\right]+8 \\
\frac{dy}{dx} &= 3\cos\left[\frac{\pi}{6}(x-4)\right]\frac{\pi}{6} \\
&= \frac{3\pi}{6}\cos\left[\frac{\pi}{6}(x-4)\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{3\pi}{6}\cos\left[\frac{\pi}{6}(x-4)\right] &= 0 \\
\cos\left[\frac{\pi}{6}(x-4)\right] &= 0 \\
\frac{\pi}{6}(x-4) &= \frac{\pi}{2}, \frac{3\pi}{2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\pi}{6}(x-4) &= \frac{\pi}{2} & \frac{\pi}{6}(x-4) &= \frac{3\pi}{2} \\
x-4 &= \frac{\pi}{2}\cdot\frac{6}{\pi} & x-4 &= \frac{3\pi}{2}\cdot\frac{6}{\pi} \\
x-4 &= 3 & x-4 &= 9 \\
x &= 7 & x &= 13
\end{aligned}
$$

Substitute 7 for x in the expression of y.

$$
\begin{aligned}
y &= 3\sin\left[\frac{\pi}{6}(x-4)\right]+8 \\
&= 3\sin\left[\frac{\pi}{6}(7-4)\right]+8 \\
&= 3\sin\left(\frac{\pi}{2}\right)+8 \\
&= 11
\end{aligned}
$$

Substitute 13 for x in the expression of y.

$$
\begin{aligned}
y &= 3\sin\left[\frac{\pi}{6}(x-4)\right]+8 \\
&= 3\sin\left[\frac{\pi}{6}(13-4)\right]+8 \\
&= 3\sin\left(\frac{3\pi}{2}\right)+8 \\
&= -3+8 \\
&= 5
\end{aligned}
$$

Find the second derivative.

$$\frac{dy}{dx} = \frac{3\pi}{6}\cos\left[\frac{\pi}{6}(x-4)\right]$$
$$\frac{d^2y}{d^2x} = -\frac{\pi}{6}\sin\left[\frac{\pi}{6}(x-4)\right]$$

Substitute 7 for *x* in the second derivative.

$$\begin{aligned}\frac{d^2y}{d^2x} &= -\frac{\pi}{6}\sin\left[\frac{\pi}{6}(7-4)\right]\\ &= -\frac{\pi}{6}\sin\left(\frac{\pi}{2}\right)\\ &= -\frac{\pi}{6} < 0\end{aligned}$$

Substitute 13 for *x* in the second derivative.

$$\begin{aligned}\frac{d^2y}{d^2x} &= -\frac{\pi}{6}\sin\left[\frac{\pi}{6}(13-4)\right]\\ &= -\frac{\pi}{6}\sin(3\pi)\\ &= \frac{\pi}{6} > 0\end{aligned}$$

The maximum depth of the River Thames on that day is 11 m.

b. The maximum depth occurs at 7.00 A.M.

Solve each equation if θ is measured in radians.

50. $(\cos\theta)(\sin 2\theta) - 2\sin\theta + 2 = 0$

SOLUTION:

$$
\begin{aligned}
(\cos\theta)(\sin 2\theta) - 2\sin\theta + 2 &= 0 \\
\cos\theta(2\sin\theta\cos\theta) - 2\sin\theta + 2 &= 0 \\
2\sin\theta\cos^2\theta - 2\sin\theta + 2 &= 0 \\
2\sin\theta\left(1 - \sin^2\theta\right) - 2\sin\theta + 2 &= 0 \\
2\sin\theta - 2\sin^3\theta - 2\sin\theta + 2 &= 0 \\
-2\sin^3\theta + 2 &= 0 \\
-2\sin^3\theta &= -2 \\
\sin^3\theta &= 1 \\
\sin\theta &= 1 \\
\theta &= \frac{\pi}{2}
\end{aligned}
$$

Period for the sine function is 2π.

So, the general solution is $\frac{\pi}{2} + 2\pi k$, where k is an integer.

13-5 Solving Trigonometric Equations

51. $2\sin^2\theta + \left(\sqrt{2}-1\right)\sin\theta = \frac{\sqrt{2}}{2}$

SOLUTION:

$$2\sin^2\theta + \left(\sqrt{2}-1\right)\sin\theta = \frac{\sqrt{2}}{2}$$
$$2\sin^2\theta + \sqrt{2}\sin\theta - \sin\theta - \frac{\sqrt{2}}{2} = 0$$
$$4\sin^2\theta + 2\sqrt{2}\sin\theta - 2\sin\theta - \sqrt{2} = 0$$
$$2\sin\theta\left(2\sin\theta + \sqrt{2}\right) - 1\left(2\sin\theta + \sqrt{2}\right) = 0$$
$$\left(2\sin\theta + \sqrt{2}\right)\left(2\sin\theta - 1\right) = 0$$
$$2\sin\theta = -\sqrt{2} \text{ or } 2\sin\theta = 1$$
$$\sin\theta = -\frac{\sqrt{2}}{2} \text{ or } \sin\theta = \frac{1}{2}$$
$$\theta = \frac{5\pi}{4}, \frac{7\pi}{4} \text{ or } \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

The general solution is $\frac{\pi}{6} + 2\pi k, \frac{5\pi}{6} + 2\pi k, \frac{5\pi}{4} + 2\pi k, \frac{7\pi}{4} + 2\pi k$, k is an integer.

Solve each equation if θ is measured in degrees.

52. $\sin 2\theta + \frac{\sqrt{3}}{2} = \sqrt{3}\sin\theta + \cos\theta$

SOLUTION:

$$\sin 2\theta + \frac{\sqrt{3}}{2} = \sqrt{3}\sin\theta + \cos\theta$$
$$2\sin\theta\cos\theta + \frac{\sqrt{3}}{2} - \sqrt{3}\sin\theta - \cos\theta = 0$$
$$4\sin\theta\cos\theta - 2\sqrt{3}\sin\theta - 2\cos\theta + \sqrt{3} = 0$$
$$2\sin\theta\left(2\cos\theta - \sqrt{3}\right) - 1\left(2\cos\theta - \sqrt{3}\right) = 0$$
$$\left(2\cos\theta - \sqrt{3}\right)\left(2\sin\theta - 1\right) = 0$$
$$2\cos\theta - \sqrt{3} = 0 \text{ or } 2\sin\theta - 1 = 0$$
$$\cos\theta = \frac{\sqrt{3}}{2} \text{ or } \sin\theta = \frac{1}{2}$$
$$\theta = 30°, 330° \text{ or } \theta = 30°, 150°$$

The general solution is $30° + 360°k, 150° + 360°k, 330° + 360°k$, k is an integer.

13-5 Solving Trigonometric Equations

53. $1-\sin^2\theta-\cos\theta=\frac{3}{4}$

SOLUTION:

$$\begin{aligned}1-\sin^2\theta-\cos\theta&=\frac{3}{4}\\ \cos^2\theta-\cos\theta-\frac{3}{4}&=0\\ 4\cos^2\theta-4\cos\theta-3&=0\\ 4\cos^2\theta-6\cos\theta+2\cos\theta-3&=0\\ 2\cos\theta(2\cos\theta-3)+1(2\cos\theta-3)&=0\\ (2\cos\theta-3)(2\cos\theta+1)&=0\\ 2\cos\theta-3=0 \text{ or } 2\cos\theta+1&=0\\ \cos\theta=\frac{3}{2} \text{ or } \cos\theta&=-\frac{1}{2}\end{aligned}$$

Since the range of the cosine function lies between –1 and 1, neglect the equation $\cos\theta=\frac{3}{2}$.

$$\begin{aligned}\cos\theta&=-\frac{1}{2}\\ \theta&=120°,\ 240°\end{aligned}$$

The general solution is $120°+360°k,\ 240°+360°k$, k is an integer.

Solve each equation.

54. $2\sin\theta=\sin 2\theta$

SOLUTION:

$$\begin{aligned}2\sin\theta&=\sin 2\theta\\ 2\sin\theta&=2\sin\theta\cos\theta\\ 2\sin\theta-2\sin\theta\cos\theta&=0\\ 2\sin\theta(1-\cos\theta)&=0\end{aligned}$$

$$\begin{aligned}2\sin\theta&=0 & 1-\cos\theta&=0\\ \sin\theta&=0 & \cos\theta&=1\\ \theta&=0,\pi & \theta&=0,3\pi\end{aligned}$$

The general solution is πk, k is an integer.

55. $\cos\theta\tan\theta - 2\cos^2\theta = -1$

SOLUTION:

$$\begin{aligned}
\cos\theta\tan\theta - 2\cos^2\theta &= -1\\
\cos\theta\frac{\sin\theta}{\cos\theta} - 2\left(1-\sin^2\theta\right) + 1 &= 0\\
\sin\theta - 2 + 2\sin^2\theta + 1 &= 0\\
2\sin^2\theta + \sin\theta - 1 &= 0\\
2\sin^2\theta + 2\sin\theta - \sin\theta - 1 &= 0\\
2\sin\theta\left(\sin\theta+1\right) - 1\left(\sin\theta+1\right) &= 0\\
\left(\sin\theta+1\right)\left(2\sin\theta-1\right) &= 0
\end{aligned}$$

$$\begin{aligned}
\sin\theta + 1 &= 0\\
\sin\theta &= -1\\
\theta &= \frac{3\pi}{2}
\end{aligned}
\qquad
\begin{aligned}
2\sin\theta - 1 &= 0\\
2\sin\theta &= 1\\
\sin\theta &= \frac{1}{2}\\
\theta &= \frac{\pi}{6}, \frac{\pi}{6}
\end{aligned}$$

$\theta = \frac{3\pi}{2}$ is the extraneous solution.

Therefore, the general solution is $\frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi$, k is an integer.

56. **DIAMONDS** According to Snell's Law $n_1 \sin i = n_2 \sin r$, where n_1 is the index of refraction of the medium the light is exiting, n_2 is the index of refraction of the medium the light is entering, *i* is the degree measure of the angle of incidence, and *r* is the degree measure of the angle of refraction.
a. The index of refraction of a diamond is 2.42, and the index of refraction of air is 1.00. If a beam of light strikes a diamond at an angle of 35°, what is the angle of refraction?
b. Explain how a gemologist might use Snell's Law to determine whether a diamond is genuine.

SOLUTION:
a. $n_1 \sin i = n_2 \sin r$

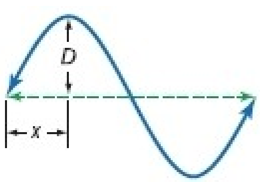
Substitute 2.42 for n_2, 1 for n_1, and 35° for *i*.

$$\begin{aligned} 1\sin 35° &= 2.42\sin r \\ \sin r &= \frac{\sin 35°}{2.42} \\ r &= \sin^{-1}\left(\frac{\sin 35°}{2.42}\right) \\ r &\approx 13.7 \end{aligned}$$

So, the angle of refraction is 13.71°.

b. Measure the angles of incidence and refraction to determine the index of refraction. If the index is 2.42, the diamond is genuine.

57. **CCSS PERSEVERANCE** A wave traveling in a guitar string can be modeled by the equation $D = 0.5\sin(6.5x)\sin(2500t)$, where *D* is the displacement in millimeters at the position *x* millimeters from the left end of the string at time *t* seconds. Find the first positive time when the point 0.5 meter from the left end has a displacement of 0.01 millimeter.



SOLUTION:
Substitute 0.01 for *D* and 0.5 for *x* in the equation $D = 0.5\sin(6.5x)\sin(2500t)$.

$$\begin{aligned} 0.01 &= 0.5\sin\left(6.5\cdot(0.5\cdot 1000)\right)\sin\left(2500t\right) \\ 0.01 &= 0.5\sin\left(3250\right)\sin\left(2500t\right) \\ \sin\left(2500t\right) &= \frac{0.01}{0.5\sin\left(3250\right)} \\ 2500t &= \text{Sin}^{-1}\left(\frac{0.01}{0.5\sin\left(3250\right)}\right) \\ t &\approx 0.0026 \text{ seconds} \end{aligned}$$

58. **MULTIPLE REPRESENTATIONS** Consider the trigonometric inequality $\sin\theta \geq \frac{1}{2}$.

a. Tabular Construct a table of values for 0° ≤ *θ* ≤ 360°. For what values of *θ* is $\sin\theta \geq \frac{1}{2}$?

b. Graphical Graph $y = \sin\theta$ and $y = \frac{1}{2}$ on the same graph for 0° ≤ *θ* ≤ 360°. For what values of *θ* is the graph of $y = \sin\theta$ above the graph of $y = \frac{1}{2}$?

c. Analytic Based on your answers for parts *a* and *b*, solve $\sin\theta \geq \frac{1}{2}$ for all values of *θ*.

d. Algebraic Solve each inequality if 0 ≤ *θ* ≤ 360°. Then solve each for all values of θ.

i. $\cos\theta \geq \frac{\sqrt{2}}{2}$

ii. $2\sin\theta \leq \sqrt{3}$

iii. $-\sin\theta \geq 0$

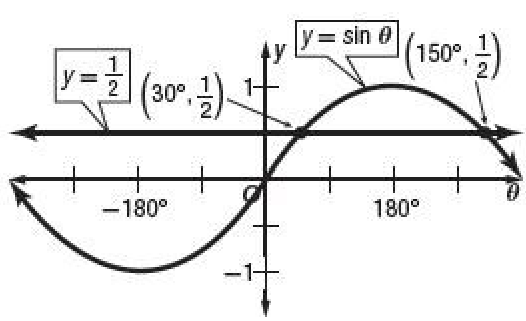
iv. $\cos\theta - 1 < -\frac{1}{2}$

SOLUTION:

a.

θ	$\sin\theta$	θ	$\sin\theta$
0°	1	210°	$-\frac{1}{2}$
30°	$\frac{1}{2}$	225°	$-\frac{\sqrt{2}}{2}$
45°	$\frac{\sqrt{2}}{2}$	240°	$-\frac{\sqrt{3}}{2}$
60°	$\frac{\sqrt{3}}{2}$	270°	−1
90°	1	300°	$-\frac{\sqrt{3}}{2}$
120°	$\frac{\sqrt{3}}{2}$	315°	$-\frac{\sqrt{2}}{2}$
135°	$\frac{\sqrt{2}}{2}$	330°	$-\frac{1}{2}$
150°	$\frac{1}{2}$	360°	0
180°	0		

b. The graph of $y = \sin\theta$ is above the graph of $y = \frac{1}{2}$ for 30° < *θ* < 150°.



c. The graph of $y = \sin\theta$ is above the graph of $y = \frac{1}{2}$ for 30° < θ < 150° and the period of sin θ is [-360°, 360°], so the solutions of $\sin\theta > \frac{1}{2}$ are 30° + k · 360° < θ < 150° + k · 360°.

d.
i. 0° ≤ θ ≤ 45° and 315° ≤ θ ≤ 360°; 0° + k · 360° ≤ θ ≤ 45° + k · 360° and 315° + k · 360° ≤ θ ≤ 360° + k · 360°
ii. 0° ≤ θ ≤ 60° and 120° ≤ θ ≤ 360°; 0° + k · 360° ≤ θ ≤ 60° + k · 360° and 120° + k · 360° ≤ θ ≤ 360° + k · 360°
iii. 180° ≤ θ ≤ 360°; 180° + k · 360° ≤ θ ≤ 360° + k · 360°
iv. 60° ≤ θ ≤ 300°; 60°+ k · 360° ≤ θ ≤ 300° + k · 360°

59. **CHALLENGE** Solve $\sin 2x < \sin x$ for $0 \le x \le 2\pi$ without a calculator.

SOLUTION:

$$\begin{aligned} \sin 2x &= \sin x \\ 2\sin x\cos x &= \sin x \\ 2\cos x &= \frac{\sin x}{\sin x} \\ 2\cos x &= 1 \\ \cos x &= \frac{1}{2} \end{aligned}$$

$$\frac{\pi}{3} < x < \pi \text{ or } \frac{5\pi}{3} < x < 2\pi$$

60. **REASONING** Compare and contrast solving trigonometric equations with solving linear and quadratic equations. What techniques are the same? What techniques are different? How many solutions do you expect?

SOLUTION:

Each type of equation may require adding, subtracting, multiplying, or dividing each side by the same number. Quadratic and trigonometric equations can often be solved by factoring. Linear and quadratic equations do not require identities. All linear and quadratic equations can be solved algebraically, whereas some trigonometric equations may be graphed more easily by using a graphing calculator. A linear equation has at most one solution. A quadratic equation has at most two solutions. A trigonometric equation usually has infinitely many solutions, unless the values of the variable are restricted.

61. **WRITING IN MATH** Why do trigonometric equations often have infinitely many solutions?

SOLUTION:

Sample answer: All trigonometric functions are periodic. Therefore, once one or more solutions are found for a certain interval, there will be additional solutions that can be found by adding integral multiples of the period of the function to those solutions.

62. **OPEN ENDED** Write an example of a trigonometric equation that has exactly two solutions if $0° \le \theta \le 360°$.

SOLUTION:

Sample answer: $2\cos\theta = 0$, 90° and 270°.

63. **CHALLENGE** How many solutions in the interval $0° \leq \theta \leq 360°$ should you expect for $a\sin(b\theta + c) + d = d\left(\frac{a}{2}\right)$, if $a \neq 0$ and b is a positive integer?

SOLUTION:

The amplitude of the graph of $y = a \sin (b\theta + c)$ is $|a|$ and the period is $\frac{360°}{b}$. Solutions to $a \sin (b\theta + c) = d$ occur when the graphs of $y = a \sin (b\theta + c)$ and $y = d$ intersect. There are three cases to consider.

Case 1: $|d| > |a|$ Because the amplitude of the graph of $y = a \sin (b\theta + c)$ is $|a|$, its maximum value is $|a|$ and its minimum is $-|a|$. In this case the graphs of $y = d$ and $y = a \sin (b\theta + c)$ will not intersect. Then there are 0 solutions to $a \sin (b\theta + c) = d$.

Case 2: $|d| = |a|$ In this case the graphs of $y = d$ and $y = a \sin (b\theta + c)$ will intersect either when $\sin (b\theta + c) = 1$ or when $\sin (b\theta + c) = -1$. This occurs once in each period of the function. In the interval 0° to 360°, there are $360° \div \frac{360°}{b}$ or b periods of the function. Thus in this case there are b solutions to $a \sin (b\theta + c) = d$.

Case 3: $|d| < |a|$ In this case the graphs of $y = d$ and $y = a \sin (b\theta + c)$ will intersect 2 times in each period of the function. Thus in this case there are $2b$ solutions to $a \sin (b\theta + c) = d$.

Therefore, there are 0, b, or $2b$ solutions in the interval $0 \leq \theta \leq 360°$.

64. **EXTENDED RESPONSE** Charles received $2500 for a graduation gift. He put it into a savings account in which the interest rate was 5.5% per year.

a. How much did he have in his savings account after 5 years if he made no deposits or withdrawals?

b. After how many years will the amount in his savings account have doubled?

SOLUTION:

a. Total amount $A = P\left(1 + \frac{R}{100}\right)^n$

Substitute 2500 for *P*, 5.5 for *R*, and 5 for n.

$$\begin{aligned} A &= P\left(1 + \frac{R}{100}\right)^n \\ &= 2500\left(1 + \frac{5.5}{100}\right)^5 \\ &= 2500(1 + 0.055)^5 \\ &= 2500(1.055)^5 \\ &= 3267.40 \end{aligned}$$

He had $3267.40 in his savings account after 5 years.

b. Substitute 5000 for A, 2500 for *P*, and 5.5 for *R* $A = P\left(1 + \frac{R}{100}\right)^n$.

$$\begin{aligned} A &= P\left(1 + \frac{R}{100}\right)^n \\ 5000 &= 2500(1 + 0.055)^n \\ 5000 &= 2500(1.055)^n \\ 2 &= (1.055)^n \\ \log 2 &= \log(1.055)^n \\ &= n \log(1.055) \\ \frac{\log 2}{\log(1.055)} &= n \\ n &\approx 13 \end{aligned}$$

After 13 years, the amount will have doubled in his savings account.

65. **PROBABILITY** Find the probability of rolling three 3s if a number cube is rolled three times.

A. $\frac{1}{216}$

B. $\frac{1}{36}$

C. $\frac{1}{6}$

D. $\frac{1}{4}$

SOLUTION:

$$\begin{aligned}\text{probability} &= \frac{\text{Number of possible outcomes}}{\text{Total number of outcome}}\\ &= \frac{1}{6^3}\\ &= \frac{1}{216}\end{aligned}$$

So, the correct option is **A**.

66. Use synthetic substitution to find $f(-2)$ for the function below.

$f(x) = x^4 + 10x^2 + x + 8$

F. 62
G. 38
H. 30
J. 8

SOLUTION:

$$\begin{aligned}f(-2) &= (-2)^4 + 10(-2)^2 + (-2) + 8\\ &= 16 + 40 - 2 + 8\\ &= 64 - 2\\ &= 62\end{aligned}$$

The correct option is **F**.

67. **SAT/ACT** The pattern of dots below continues infinitely, with more dots being added at each step.

Which expression can be used to determine the number of dots in the *n*th step?
A $2n$
B $n(n + 2)$
C $n(n + 1)$
D $2(n + 2)$
E $2(n + 1)$

SOLUTION:
4, 6, 8,...
The common difference is 2.
So, the *n*th term of the sequence is $2n$.
Each term is 2 more than $2n$.
So, an expression that can be used to find the *n*th term is $2n + 2$ or $2(n + 1)$.
Therefore, the correct option is E.

Find the exact value of each expression.

68. cos 165°

SOLUTION:

$$\begin{aligned}\cos 165^\circ &= \cos\frac{330^\circ}{2}\\ &= \pm\sqrt{\frac{1+\cos 330^\circ}{2}}\end{aligned}$$

Find the value of $\cos 330^\circ$.

$$\begin{aligned}\cos 330^\circ &= \cos(360^\circ - 30^\circ)\\ &= \cos 30^\circ\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

$$\begin{aligned}\cos\frac{330^\circ}{2} &= \pm\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}\\ &= \pm\sqrt{\frac{2+\sqrt{3}}{4}}\\ &= \pm\sqrt{\frac{2+\sqrt{3}}{4}}\\ &= \pm\frac{\sqrt{2+\sqrt{3}}}{2}\end{aligned}$$

Since $\theta = 165^\circ$ is lies in the II quadrant,

$\cos 165^\circ = \frac{\sqrt{2+\sqrt{3}}}{2}$.

13-5 Solving Trigonometric Equations

69. $\sin 22\frac{1}{2}°$

SOLUTION:

$$\begin{aligned}\sin 22\frac{1}{2}° &= \sin\frac{45°}{2}\\ &= \pm\sqrt{\frac{1-\cos 45°}{2}}\\ &= \pm\sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}}\\ &= \pm\sqrt{\frac{\sqrt{2}-1}{4}}\\ &= \pm\frac{\sqrt{\sqrt{2}-1}}{2}\\ &= \pm\frac{\sqrt{\sqrt{2}\left(\sqrt{2}-1\right)}}{2}\\ &= \pm\frac{\sqrt{2-\sqrt{2}}}{2}\end{aligned}$$

Since $\theta = 22\frac{1}{2}°$ is lies in the I quadrant,

$\sin 22\frac{1}{2}° = \frac{\sqrt{2-\sqrt{2}}}{2}$.

70. $\sin\frac{7\pi}{8}$

SOLUTION:

$$\begin{aligned}\sin\frac{7\pi}{8} &= \sin 157.5°\\ &= \sin\frac{315°}{2}\\ &= \pm\sqrt{\frac{1-\cos 315°}{2}} =\end{aligned}$$

Find the value of $\cos 315°$.

$$\begin{aligned}\cos 315° &= \cos\left(360-45°\right)\\ &= \cos 45°\\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

$$\sin\frac{315°}{2} = \pm\sqrt{\frac{1-\cos 315°}{2}}$$
$$= \pm\sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}}$$
$$= \pm\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$$
$$= \pm\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}\times\frac{2\sqrt{2}}{2\sqrt{2}}}$$
$$= \pm\sqrt{\frac{2\sqrt{2}\left(\sqrt{2}-1\right)}{2\sqrt{2}\left(2\sqrt{2}\right)}}$$
$$= \pm\sqrt{\frac{2\left(2-\sqrt{2}\right)}{8}}$$
$$= \pm\sqrt{\frac{\left(2-\sqrt{2}\right)}{4}}$$
$$= \pm\frac{\sqrt{\left(2-\sqrt{2}\right)}}{2}$$

Since $\theta = \frac{7\pi}{8}$ is lies in the II quadrant,

$$\sin\frac{7\pi}{8} = \frac{\sqrt{\left(2-\sqrt{2}\right)}}{2}$$

13-5 Solving Trigonometric Equations

71. $\cos\frac{7\pi}{12}$

SOLUTION:

$$\cos\frac{7\pi}{12} = \cos 75° = \cos\frac{150°}{2}$$

Use the half angle formula of cosine.

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$
$$\cos\frac{150°}{2} = \pm\sqrt{\frac{1+\cos 150°}{2}}$$

$$\begin{aligned}\cos 150° &= \cos(90° + 60°)\\ &= -\sin 60°\\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

Substitute $-\frac{\sqrt{3}}{2}$ for $\cos 150°$.

$$\begin{aligned}\cos\frac{150°}{2} &= \pm\sqrt{\frac{1+\cos 150°}{2}}\\ &= \pm\sqrt{\frac{1+\left(-\frac{\sqrt{3}}{2}\right)}{2}}\\ &= \pm\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}\\ &= \pm\sqrt{\frac{2-\sqrt{3}}{4}}\\ &= \pm\frac{\sqrt{2-\sqrt{3}}}{2}\end{aligned}$$

Since $\theta = 150°$ is lies in the II quadrant,
$\cos\frac{150°}{2} = -\frac{\sqrt{2-\sqrt{3}}}{2}$.

Verify that each equation is an identity.

72. $\sin(270° - \theta) = -\cos\theta$

SOLUTION:

$\sin(270° - \theta) \stackrel{?}{=} -\cos\theta$

$\sin 270° \cos\theta - \cos 270° \sin\theta \stackrel{?}{=} -\cos\theta$

$-1\cos\theta - 0 \stackrel{?}{=} -\cos\theta$

$-\cos\theta = -\cos\theta$ ✓

73. $\cos(90° + \theta) = -\sin\theta$

SOLUTION:

$\cos(90° + \theta) \stackrel{?}{=} -\sin\theta$

$\cos 90° \cos\theta - \sin 90° \sin\theta \stackrel{?}{=} -\sin\theta$

$0 - 1\sin\theta - 0 \stackrel{?}{=} -\sin\theta$

$-\sin\theta = -\sin\theta$ ✓

74. $\cos(90° - \theta) = \sin\theta$

SOLUTION:

$\cos(90° - \theta) \stackrel{?}{=} \sin\theta$

$\cos 90° \cos\theta + \sin 90° \sin\theta \stackrel{?}{=} \sin\theta$

$0 \cdot \cos\theta + 1 \cdot \sin\theta - 0 \stackrel{?}{=} \sin\theta$

$\sin\theta = \sin\theta$ ✓

75. $\sin(90° - \theta) = \cos\theta$

SOLUTION:

$$\sin(90° - \theta) \stackrel{?}{=} \cos\theta$$

$$\sin 90° \cos\theta - \cos 90° \sin\theta \stackrel{?}{=} \cos\theta$$

$$1 \cdot \cos\theta - 0 \cdot \sin\theta \stackrel{?}{=} \cos\theta$$

$$\cos\theta - 0 \stackrel{?}{=} \cos\theta$$

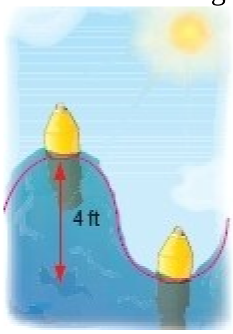
$$\cos\theta = \cos\theta \checkmark$$

76. **WATER SAFETY** A harbor buoy bobs up and down with the waves. The distance between the highest and lowest points is 4 feet. The buoy moves from its highest point to its lowest point and back to its highest point every 10 seconds.

a. Write an equation for the motion of the buoy. Assume that it is at equilibrium at $t = 0$ and that it is on the way up from the normal water level.

b. Draw a graph showing the height of the buoy as a function of time.

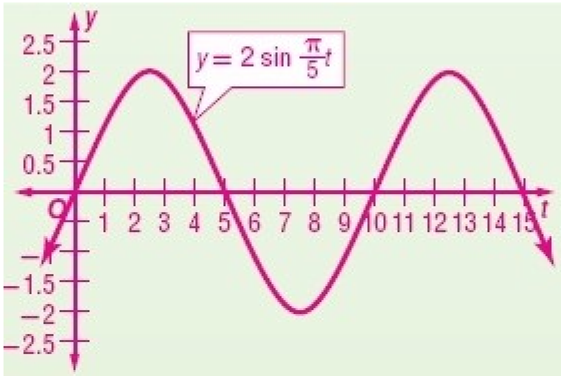
c. What is the height of the buoy after 12 seconds?



SOLUTION:

a. $y = 2\sin\frac{\pi}{5}t$

b.



c. Substitute 12 fot *t* in the equation $y = 2\sin\frac{\pi}{5}t$.

$$\begin{aligned} y &= 2\sin\frac{\pi}{5}(12) \\ &= 2\sin\frac{12\pi}{5} \\ &\approx 1.9 \end{aligned}$$

After 12 seconds, the height of the buoy is about 1.9 feet.

Find the first three terms of each arithmetic series described.

77. $a_1 = 17, a_n = 197, S_n = 2247$

SOLUTION:

Substitute 2247 for S_n, 17 for a_1, and 197 for a_n

$$S_n = \frac{n(a_1 + a_n)}{2}$$
$$2247 = \frac{n(17 + 197)}{2}$$
$$2247 = \frac{n(214)}{2}$$
$$2247 = n(107)$$
$$n = 21$$

Find d.

$$a_n = a_1 + (n-1)d$$
$$197 = 17 + (21-1)d$$
$$197 = 17 + 20d$$
$$d = 9$$

Find a_2 and a_3.

$a_2 = a_1 + 9 = 17 + 9 = 26$

$a_3 = a_2 + 9 = 26 + 9 = 35$

Therefore, the first three terms are 17, 26, 35.

78. $a_1 = -13, a_n = 427, S_n = 18,423$

SOLUTION:

Substitute 18,423 for S_n, –13 for a_1, and 427 for a_n.

$$\begin{aligned} S_n &= \frac{n(a_1 + a_n)}{2} \\ 18,423 &= \frac{n(-13 + 427)}{2} \\ 18,423 &= \frac{n(414)}{2} \\ 18,423 &= n(207) \\ n &= 89 \end{aligned}$$

Find d.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ 427 &= -13 + (89-1)d \\ 427 &= -13 + 88d \\ 440 &= 88d \\ d &= 5 \end{aligned}$$

Find a_2 and a_3.

$a_2 = a_1 + 5 = -13 + 5 = -8$

$a_3 = a_2 + 5 = -8 + 5 = -3$

Therefore, the first three terms are –13, –8, –3.

79. $n = 31, a_n = 78, S_n = 1023$

SOLUTION:

Substitute 1023 for S_n, 31 for n, and 78 for a_n.

$$
\begin{aligned}
S_n &= \frac{n(a_1 + a_n)}{2} \\
1023 &= \frac{31(a_1 + 78)}{2} \\
2046 &= 31(a_1 + 78) \\
66 &= (a_1 + 78) \\
a_1 &= 66 - 78 \\
a_1 &= -12
\end{aligned}
$$

$a_n = a_1 + (n - 1)d$

Find d.

$$
\begin{aligned}
a_n &= a_1 + (n-1)d \\
78 &= -12 + (31-1)d \\
78 &= -12 + 30d \\
90 &= 30d \\
d &= 3
\end{aligned}
$$

Find a_2 and a_3.

$a_2 = a_1 + 3 = -12 + 3 = -9$

$a_3 = a_2 + 3 = -9 + 3 = -6$

Therefore, the first three terms are –12, –9, –6.

80. $n = 19, a_n = 103, S_n = 1102$

SOLUTION:

Substitute 1102 for S_n, 19 for n, and 103 for a_n.

$$S_n = \frac{n(a_1 + a_n)}{2}$$
$$1102 = \frac{19(a_1 + 103)}{2}$$
$$2204 = 19(a_1 + 103)$$
$$116 = (a_1 + 103)$$
$$a_1 = 116 - 103$$
$$a_1 = 13$$

Find d.

$$a_n = a_1 + (n-1)d$$
$$103 = 13 + (19-1)d$$
$$103 = 13 + 18d$$
$$90 = 18d$$
$$d = 5$$

Find a_2 and a_3.

$a_2 = a_1 + 5 = 13 + 5 = 18$

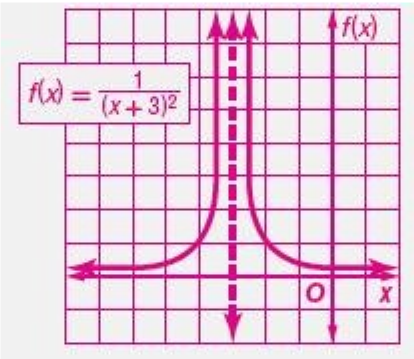
$a_3 = a_2 + 5 = 18 + 5 = 23$

Therefore, the first three terms are 13, 18, 23.

Graph each rational function.

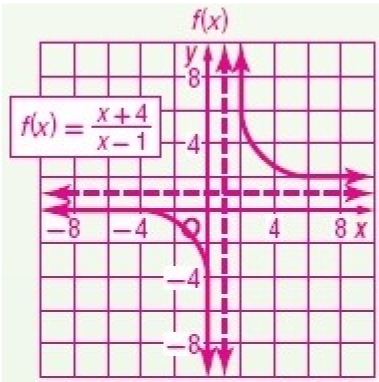
81. $f(x) = \dfrac{1}{(x+3)^2}$

SOLUTION:



82. $f(x) = \dfrac{x+4}{x-1}$

SOLUTION:



83. $f(x) = \dfrac{x+2}{x^2-x-6}$

SOLUTION: